Deptt- MATHEMATICS

Topic- Indefinite Integration

College- SOGHRA COLLEGE,BIHAR SHARIF

Part- BSc PART 1

INDEFINITE INTEGRATION

If f & F are function of x such that F' (x) = f(x) then the function F is called a **PRIMITIVE OR ANTIDERIVATIVE OR INTEGRAL** of f(x) w.r.t. x and is written symbolically as

$\int f(x)\,dx = F(x) + c \Leftrightarrow \frac{d}{dx}\{F(x) + c\} = f(x)$, where c is called the **constant of integration.**

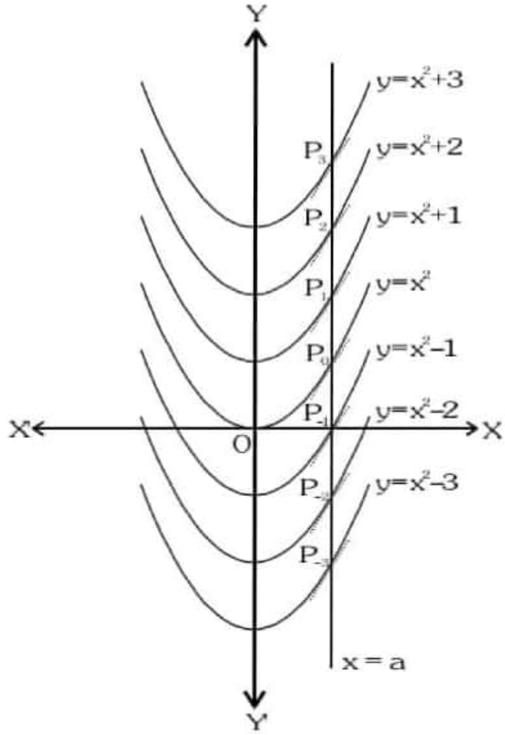
1. GEOMETRICAL INTERPRETATION OF INDEFINITE INTEGRAL :

$\int f(x)dx = F(x) + c = y(\text{say})$, represents a family of curves. The different values of c will correspond to different members of this family and these members can be obtained by shifting any one of the curves parallel to itself. This is the geometrical interpretation of indefinite integral.

Let f(x) = 2x. Then $\int f(x)dx = x^2 + c$. For different values of c, we get different integrals. But these integrals are very similar geometrically.

Thus, $y = x^2 + c$, where c is arbitrary constant, represents a family of integrals. By assigning different values to c, we get different members of the family. These together constitute the indefinite integral. In this case, each integral represents a parabola with its axis along y-axis.

If the line x = a intersects the parabolas $y = x^2$, $y = x^2 + 1$, $y = x^2 + 2$, $y = x^2 - 1$, $y = x^2 - 2$ at P_0, P_1, P_2, P_{-1}, P_{-2} etc., respectively, then $\frac{dy}{dx}$ at these points equals 2a. This indicates that the tangents to the curves at these points are parallel. Thus, $\int 2x\,dx = x^2 + c = f(x) + c$ **(say), implies that the tangents to all the curves f(x) + c, c ∈ R, at the points of intersection of the curves by the line x = a, (a ∈ R) , are parallel.**



2. STANDARD RESULTS :

(i) $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c;\ n \neq -1$

(ii) $\int \frac{dx}{ax+b} = \frac{1}{a}\ell n\,|ax+b| + c$

(iii) $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + c$

(iv) $\int a^{px+q} dx = \frac{1}{p}\frac{a^{px+q}}{\ell n a} + c,\ (a > 0)$

(v) $\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + c$

(vi) $\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + c$

(vii) $\int \tan(ax+b)dx = \frac{1}{a}\ell n\,|\sec(ax+b)| + c$

(viii) $\int \cot(ax+b)dx = \frac{1}{a}\ell n\,|\sin(ax+b)| + c$

(ix) $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b) + c$

(x) $\int \operatorname{cosec}^2(ax+b)dx = -\frac{1}{a}\cot(ax+b) + c$

(xi) $\int \operatorname{cosec}(ax+b).\cot(ax+b)dx = -\frac{1}{a}\operatorname{cosec}(ax+b) + c$

(xii) $\int \sec(ax+b).\tan(ax+b)dx = \frac{1}{a}\sec(ax+b) + c$

(xiii) $\int \sec x\, dx = \ell n|\sec x + \tan x| + c = \ell n\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + c$

(xiv) $\int \text{cosec}\, x\, dx = \ell n|\text{cosec}\, x - \cot x| + c = \ell n\left|\tan\frac{x}{2}\right| + c = -\ell n\,|\,\text{cosec}\, x + \cot x\,| + c$

(xv) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c$

(xvi) $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

(xvii) $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + c$

(xviii) $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ell n\left[x + \sqrt{x^2 + a^2}\right] + c$

(xix) $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ell n\left[x + \sqrt{x^2 - a^2}\right] + c$

(xx) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ell n\left|\frac{a + x}{a - x}\right| + c$

(xxi) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ell n\left|\frac{x - a}{x + a}\right| + c$

(xxii) $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$

(xxiii) $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ell n\left(x + \sqrt{x^2 + a^2}\right) + c$

(xxiv) $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ell n\left(x + \sqrt{x^2 - a^2}\right) + c$

(xxv) $\int e^{ax}.\sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a\sin bx - b\cos bx) + c = \frac{e^{ax}}{\sqrt{a^2 + b^2}}\sin\left(bx - \tan^{-1}\frac{b}{a}\right) + c$

(xxvi) $\int e^{ax}.\cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a\cos bx + b\sin bx) + c = \frac{e^{ax}}{\sqrt{a^2 + b^2}}\cos\left(bx - \tan^{-1}\frac{b}{a}\right) + c$

3. TECHNIQUES OF INTEGRATION :

(a) Substitution or change of independent variable :

If $\phi(x)$ is a continuous differentiable function, then to evaluate integrals of the form $\int f(\phi(x))\phi'(x)dx$, we substitute $\phi(x) = t$ and $\phi'(x)dx = dt$.

Hence $I = \int f(\phi(x))\phi'(x)dx$ reduces to $\int f(t)dt$.

(i) Fundamental deductions of method of substitution :

$\int [f(x)]^n f'(x)dx$ OR $\int \frac{f'(x)}{[f(x)]^n}\, dx$ put $f(x) = t$ & proceed.

Illustration 1 : Evaluate $\int \frac{\cos^3 x}{\sin^2 x + \sin x}dx$

Solution :

$I = \int \frac{(1 - \sin^2 x)\cos x}{\sin x(1 + \sin x)}dx = \int \frac{1 - \sin x}{\sin x}\cos x\, dx$

Put $\sin x = t \Rightarrow \cos x\, dx = dt$

$\Rightarrow I = \int \frac{1 - t}{t}dt = \ell n\,|\,t\,| - t + c = \ell n\,|\sin x| - \sin x + c$ **Ans.**

Illustration 2 : Evaluate $\int \frac{(x^2 - 1)dx}{(x^4 + 3x^2 + 1)\tan^{-1}\left(x + \frac{1}{x}\right)}$

Solution : The given integral can be written as

$$I = \int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left[\left(x+\frac{1}{x}\right)^2+1\right]\tan^{-1}\left(x+\frac{1}{x}\right)}$$

Let $\left(x+\frac{1}{x}\right) = t$. Differentiating we get $\left(1-\frac{1}{x^2}\right)dx = dt$

Hence $I = \int \frac{dt}{(t^2+1)\tan^{-1}t}$

Now make one more substitution $\tan^{-1}t = u$. Then $\frac{dt}{t^2+1} = du$ and $I = \int \frac{du}{u} = \ell n|u| + c$

Returning to t, and then to x, we have

$I = \ell n|\tan^{-1}t| + c = \ell n\left|\tan^{-1}\left(x+\frac{1}{x}\right)\right| + c$ **Ans.**

Do yourself -1 :

(i) Evaluate : $\int \frac{x^2}{9+16x^6}dx$ **(ii)** Evaluate : $\int \cos^3 x\, dx$

(ii) Standard substitutions :

$\int \frac{dx}{a^2+x^2}$ or $\int \sqrt{a^2+x^2}\,dx$; put $x = a\tan\theta$ or $x = a\cot\theta$

$\int \frac{dx}{a^2-x^2}$ or $\int \sqrt{a^2-x^2}\,dx$; put $x = a\sin\theta$ or $x = a\cos\theta$

$\int \frac{dx}{\sqrt{x^2-a^2}}$ or $\int \sqrt{x^2-a^2}\,dx$; put $x = a\sec\theta$ or $x = a\,\mathrm{cosec}\,\theta$

$\int \sqrt{\frac{a-x}{a+x}}\,dx$; put $x = a\cos 2\theta$

$\int \sqrt{\frac{x-\alpha}{\beta-x}}\,dx$ or $\int \sqrt{(x-\alpha)(\beta-x)}$; put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

$\int \sqrt{\frac{x-\alpha}{x-\beta}}\,dx$ or $\int \sqrt{(x-\alpha)(x-\beta)}$; put $x = \alpha\sec^2\theta - \beta\tan^2\theta$

$\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}$; put $x - \alpha = t^2$ or $x - \beta = t^2$.

Illustration 3 : Evaluate $\int \frac{dx}{\sqrt{(x-a)(b-x)}}$

Solution : Put $x = a\cos^2\theta + b\sin^2\theta$, the given integral becomes

$$I=\int \frac{2(b-a)\sin\theta\cos\theta d\theta}{\{(a\cos^2\theta+b\sin^2\theta-a)(b-a\cos^2\theta-b\sin^2\theta)\}^{\frac{1}{2}}}$$

$$=\int \frac{2(b-a)\sin\theta\cos\theta d\theta}{(b-a)\sin\theta\cos\theta}=\left(\frac{b-a}{b-a}\right)\int 2d\theta=2\theta+c=2\sin^{-1}\sqrt{\left(\frac{x-a}{b-a}\right)}+c$$ **Ans.**

Illustration 4 : Evaluate $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}.\frac{1}{x}dx$

Solution : Put $x=\cos^2\theta \Rightarrow dx=-2\sin\theta\cos\theta\, d\theta$

$$\Rightarrow \quad I=\int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}.\frac{1}{\cos^2\theta}(-2\sin\theta\cos\theta)d\theta=-\int 2\tan\frac{\theta}{2}\tan\theta\, d\theta$$

$$=-4\int \frac{\sin^2(\theta/2)}{\cos\theta}d\theta=-2\int \frac{1-\cos\theta}{\cos\theta}d\theta=-2\ell n\,|\sec\theta-\tan\theta|+2\theta+c$$

$$=-2\ell n\left|\frac{1+\sqrt{1-x}}{x}\right|+2\cos^{-1}\sqrt{x}+c$$

Do yourself -2 :

(i) Evaluate : $\int \sqrt{\frac{x-3}{2-x}}dx$ **(ii)** Evaluate : $\int \frac{dx}{x\sqrt{x^2+4}}$

(b) **Integration by part :** $\int u.v\,dx=u\int v\,dx-\int\left[\frac{du}{dx}.\int v\,dx\right]dx$ where u & v are differentiable functions and are commonly designated as first & second function respectively.

Note : While using integration by parts, choose u & v such that

(i) $\int v dx$ & (ii) $\int\left[\frac{du}{dx}.\int v\,dx\right]dx$ are simple to integrate.

This is generally obtained by choosing first function as the function which comes first in the word **ILATE**, where; I-Inverse function, L-Logarithmic function, A-Algebraic function, T-Trigonometric function & E-Exponential function.

Illustration 5 : Evaluate : $\int \cos\sqrt{x}\,dx$

Solution : Consider $I=\int \cos\sqrt{x}dx$

Let $\sqrt{x}=t$ then $\frac{1}{2\sqrt{x}}dx=dt$

i.e. $dx=2\sqrt{x}dt$ or $dx = 2t\, dt$

so $I=\int \cos t.2tdt$

taking t as first function, then integrate it by part

$$\Rightarrow \quad I=2\left[t\int \cos tdt-\int\left\{\frac{dt}{dt}\int \cos tdt\right\}dt\right]=2\left[t\sin t-\int 1.\sin tdt\right]=2[t\sin t+\cos t]+c$$

$I=2[\sqrt{x}\sin\sqrt{x}+\cos\sqrt{x}]+c$ **Ans.**

Illustration 6 : Evaluate : $\int \frac{x}{1+\sin x}dx$

Solution : Let $I = \int \frac{x}{1+\sin x}dx = \int \frac{x(1-\sin x)}{(1+\sin x)(1-\sin x)}dx$

$$= \int \frac{x(1-\sin x)}{1-\sin^2 x}dx = \int \frac{x(1-\sin x)}{\cos^2 x}dx = \int x\sec^2 x dx - \int x \sec x \tan x dx$$

$$= \left[x\int \sec^2 x dx - \int\left\{\frac{dx}{dx}\int \sec^2 x dx\right\}dx\right] - \left[x\int \sec x \tan x dx - \int\left\{\frac{dx}{dx}\int \sec x \tan x dx\right\}dx\right]$$

$$= \left[x\tan x - \int \tan x dx\right] - \left[x \sec x - \int \sec x dx\right]$$

$$= [x\tan x - \ell n|\sec x|] - [x\sec x - \ell n|\sec x + \tan x|] + c$$

$$= x(\tan x - \sec x) + \ell n\left|\frac{(\sec x + \tan x)}{\sec x}\right| + c = \frac{-x(1-\sin x)}{\cos x} + \ell n|1+\sin x| + c$$ **Ans.**

Do yourself -3 :

(i) Evaluate : $\int xe^x dx$ **(ii)** Evaluate : $\int x^3 \sin(x^2)dx$

Two classic integrands :

(i) $\int e^x[f(x) + f'(x)]dx = e^x.f(x) + c$

Illustration 7 : Evaluate $\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx$

Solution : $\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx = \int e^x \frac{(1-2x+x^2)}{(1+x^2)^2}dx = \int e^x\left(\frac{1}{(1+x^2)} - \frac{2x}{(1+x^2)^2}\right)dx = \frac{e^x}{1+x^2} + c$ **Ans.**

Illustration 8 : The value of $\int e^x\left(\frac{x^4+2}{(1+x^2)^{5/2}}\right)dx$ is equal to -

(A) $\frac{e^x(x+1)}{(1+x^2)^{3/2}}$ (B) $\frac{e^x(1-x+x^2)}{(1+x^2)^{3/2}}$ (C) $\frac{e^x(1-x)}{(1+x^2)^{3/2}}$ (D) none of these

Solution : Let $I = \int e^x\left(\frac{x^4+2}{(1+x^2)^{5/2}}\right)dx = \int e^x\left(\frac{1}{(1+x^2)^{1/2}} + \frac{1-2x^2}{(1+x^2)^{5/2}}\right).dx$

$$= \int e^x\left(\frac{1}{(1+x^2)^{1/2}} - \frac{x}{(1+x^2)^{3/2}} + \frac{x}{(1+x^2)^{3/2}} + \frac{1-2x^2}{(1+x^2)^{5/2}}\right)dx$$

$$= \frac{e^x}{(1+x^2)^{1/2}} + \frac{xe^x}{(1+x^2)^{3/2}} + c = \frac{e^x\{1+x^2+x\}}{(1+x^2)^{3/2}} + c$$ **Ans. (D)**

Do yourself -4 :

(i) Evaluate : $\int e^x\left(\tan^{-1}x + \frac{1}{1+x^2}\right)dx$ **(ii)** Evaluate : $\int xe^{x^2}\left(\sin x^2 + \cos x^2\right)dx$

(ii) $\int [f(x) + xf'(x)]dx = x f(x) + c$

Illustration 9 : Evaluate $\int \frac{x+\sin x}{1+\cos x}dx$

Solution : $I = \int \frac{x + \sin x}{1 + \cos x} dx = \int \left(\frac{x + \sin x}{2\cos^2 \frac{x}{2}} \right) dx = \int \left(x \frac{1}{2} \sec^2 \frac{x}{2} + \tan \frac{x}{2} \right) dx = x \tan \frac{x}{2} + c$ **Ans.**

Do yourself -5 :

(i) Evaluate : $\int \left(\tan(e^x) + xe^x \sec^2(e^x) \right) dx$ **(ii)** Evaluate : $\int (\ell nx + 1) dx$

(c) Integration of rational function :

(i) Rational function is defined as the ratio of two polynomials in the form $\frac{P(x)}{Q(x)}$, where P(x) and Q(x) are polynomials in x and $Q(x) \neq 0$. If the degree of P(x) is less than the degree of Q(x), then the rational function is called proper, otherwise, it is called improper. The improper rational function can be reduced to the proper rational functions by long division process. Thus, if $\frac{P(x)}{Q(x)}$ is improper, then $\frac{P(x)}{Q(x)} = T(x) + \frac{P_1(x)}{Q(x)}$, where T(x) is a polynomial in x and $\frac{P_1(x)}{Q(x)}$ is proper rational function. It is always possible to write the integrand as a sum of simpler rational functions by a method called partial fraction decomposition. After this, the integration can be carried out easily using the already known methods.

S. No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px^2 + qx + r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
2.	$\frac{px^2 + qx + r}{(x-a)^2 (x-b)}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
3.	$\frac{px^2 + qx + r}{(x-a)(x^2 + bx + c)}$ where $x^2 + bx + c$ cannot be factorised further	$\frac{A}{x-a} + \frac{Bx + C}{x^2 + bx + c}$
4.	$\frac{f(x)}{(x-a)(x^2 + bx + c)^2}$ where f(x) is a polynomial of degree less than 5.	$\frac{A}{x-a} + \frac{Bx + C}{x^2 + bx + c} + \frac{Dx + E}{(x^2 + bx + c)^2}$

Illustration 10 : Evaluate $\int \frac{x}{(x-2)(x+5)} dx$

Solution : $\frac{x}{(x-2)(x+5)} = \frac{A}{x-2} + \frac{B}{x+5}$

or $x = A(x + 5) + B(x - 2)$.

by comparing the coefficients, we get

A = 2/7 and B = 5/7 so that

$\int \frac{x}{(x-2)(x+5)} dx = \frac{2}{7} \int \frac{dx}{x-2} + \frac{5}{7} \int \frac{dx}{x+5} = \frac{2}{7} \ell n|(x-2)| + \frac{5}{7} \ell n|(x+5)| + c$ **Ans.**

Illustration 11 : Evaluate $\int \frac{x^4}{(x+2)(x^2+1)}dx$

Solution : $\frac{x^4}{(x+2)(x^2+1)} = (x-2) + \frac{3x^2+4}{(x+2)(x^2+1)}$

Now, $\frac{3x^2+4}{(x+2)(x^2+1)} = \frac{16}{5(x+2)} + \frac{-\frac{1}{5}x+\frac{2}{5}}{x^2+1}$

So, $\frac{x^4}{(x+2)(x^2+1)} = x-2+\frac{16}{5(x+2)} + \frac{-\frac{1}{5}x+\frac{2}{5}}{x^2+1}$

Now, $\int\left((x-2)+\frac{16}{5(x+2)}+\frac{-\frac{1}{5}x+\frac{2}{5}}{x^2+1}\right)dx$

$= \frac{x^2}{2} - 2x + \frac{2}{5}\tan^{-1}x + \frac{16}{5}\ell n|x+2| - \frac{1}{10}\ell n(x^2+1) + c$ **Ans.**

Do yourself - 6 :

(i) Evaluate : $\int \frac{3x+2}{(x+1)(x+3)}dx$ **(ii)** Evaluate : $\int \frac{x^2-1}{(x+1)(x+2)}dx$

(ii) $\int \frac{dx}{ax^2+bx+c}, \int \frac{dx}{\sqrt{ax^2+bx+c}}, \int \sqrt{ax^2+bx+c}\,dx$

Express $ax^2 + bx + c$ in the form of perfect square & then apply the standard results.

(iii) $\int \frac{px+q}{ax^2+bx+c}dx, \int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$

Express $px + q = \ell$ (differential coefficient of denominator) + m.

Illustration 12 : Evaluate $\int \frac{dx}{2x^2+x-1}$

Solution : $I = \int \frac{dx}{2x^2+x-1} = \frac{1}{2}\int \frac{dx}{x^2+\frac{x}{2}-\frac{1}{2}} = \frac{1}{2}\int \frac{dx}{x^2+\frac{x}{2}+\frac{1}{16}-\frac{1}{16}-\frac{1}{2}}$

$= \frac{1}{2}\int \frac{dx}{(x+1/4)^2-9/16} = \frac{1}{2}\int \frac{dx}{(x+1/4)^2-(3/4)^2}$

$= \frac{1}{2}\cdot\frac{1}{2(3/4)}\log\left|\frac{x+1/4-3/4}{x+1/4+3/4}\right| + c$ $\left\{\text{using}, \int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c\right\}$

$= \frac{1}{3}\log\left|\frac{x-1/2}{x+1}\right| + c = \frac{1}{3}\log\left|\frac{2x-1}{2(x+1)}\right| + c$ **Ans.**

Illustration 13 : Evaluate $\int \frac{3x+2}{4x^2+4x+5}dx$

Solution : Express $3x + 2 = \ell(\text{d.c. of } 4x^2 + 4x + 5) + m$

or, $3x + 2 = \ell(8x + 4) + m$

Comparing the coefficients, we get

$8\ell = 3$ and $4\ell + m = 2 \Rightarrow \ell = 3/8$ and $m = 2 - 4\ell = 1/2$

$$\Rightarrow I = \frac{3}{8}\int\frac{8x+4}{4x^2+4x+5}dx + \frac{1}{2}\int\frac{dx}{4x^2+4x+5}$$

$$= \frac{3}{8}\log\left|4x^2+4x+5\right| + \frac{1}{8}\int\frac{dx}{x^2+x+\frac{5}{4}} = \frac{3}{8}\log\left|4x^2+4x+5\right| + \frac{1}{8}\tan^{-1}\left(x+\frac{1}{2}\right)+c$$ **Ans.**

Do yourself -7 :

(i) Evaluate : $\int\frac{dx}{x^2+x+1}$ **(ii)** Evaluate : $\int\frac{5x+4}{\sqrt{x^2+3x+2}}dx$

(iv) Integrals of the form $\int\frac{x^2+1}{x^4+Kx^2+1}dx$ OR $\int\frac{x^2-1}{x^4+Kx^2+1}dx$ where K is any constant.

Divide N^r & D^r by x^2 & proceed.

Note : Sometimes it is useful to write the integral as a sum of two related integrals, which can be evaluated by making suitable substitutions e.g.

* $\int\frac{2x^2}{x^4+1}dx = \int\frac{x^2+1}{x^4+1}dx + \int\frac{x^2-1}{x^4+1}dx$ * $\int\frac{2}{x^4+1}dx = \int\frac{x^2+1}{x^4+1}dx - \int\frac{x^2-1}{x^4+1}dx$

These integrals can be called as **Algebric Twins**.

Illustration 14 : Evaluate : $\int\frac{4}{\sin^4 x+\cos^4 x}dx$

Solution : $I = 4\int\frac{1}{\sin^4 x+\cos^4 x}dx = 4\int\frac{\sin^2 x+\cos^2 x}{\sin^4 x+\cos^4 x}dx$

$$= 4\int\frac{(\tan^2 x+1)\cos^2 x}{(\tan^4 x+1)\cos^4 x}dx = 4\int\frac{(\tan^2 x+1)\sec^2 x}{(\tan^4 x+1)}dx$$

Now, put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$\Rightarrow I = 4\int\frac{1+t^2}{1+t^4}dt = 4\int\frac{1/t^2+1}{t^2+1/t^2}dt$$

Now, put $t - 1/t = z \Rightarrow \left(1+\frac{1}{t^2}\right)dt = dz$

$$\Rightarrow I = 4\int\frac{dz}{z^2+2} = \frac{4}{\sqrt{2}}\tan^{-1}\frac{z}{\sqrt{2}} = 2\sqrt{2}\tan^{-1}\frac{t-1/t}{\sqrt{2}} = 2\sqrt{2}\tan^{-1}\left(\frac{\tan x-1/\tan x}{\sqrt{2}}\right)+c$$ **Ans.**

Illustration 15 : Evaluate : $\int\frac{1}{x^4+5x^2+1}dx$

Solution : $I = \frac{1}{2}\int \frac{2}{x^4+5x^2+1}dx$

$\Rightarrow \quad I = \frac{1}{2}\int \frac{1+x^2}{x^4+5x^2+1}dx + \frac{1}{2}\int \frac{1-x^2}{x^4+5x^2+1}dx = \frac{1}{2}\int \frac{1+1/x^2}{x^2+5+1/x^2}dx - \frac{1}{2}\int \frac{1-1/x^2}{x^2+5+1/x^2}dx$

{dividing N^r and D^r by x^2}

$= \frac{1}{2}\int \frac{(1+1/x^2)}{(x-1/x)^2+7}dx - \frac{1}{2}\int \frac{(1-1/x^2)dx}{(x+1/x)^2+3} = \frac{1}{2}\int \frac{dt}{t^2+(\sqrt{7})^2} - \frac{1}{2}\int \frac{du}{u^2+(\sqrt{3})^2}$

where $t = x - \frac{1}{x}$ and $u = x + \frac{1}{x}$

$I = \frac{1}{2}.\frac{1}{\sqrt{7}}\left(\tan^{-1}\frac{t}{\sqrt{7}}\right) - \frac{1}{2}.\frac{1}{\sqrt{3}}\left(\tan^{-1}\frac{u}{\sqrt{3}}\right) + c$

$= \frac{1}{2}\left[\frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{x-1/x}{\sqrt{7}}\right) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x+1/x}{\sqrt{3}}\right)\right] + c$ **Ans.**

Do yourself -8 :

(i) Evaluate : $\int \frac{x^2+1}{x^4-x^2+1}dx$ **(ii)** Evaluate : $\int \frac{1}{1+x^4}dx$

(d) Manipulating integrands :

(i) $\int \frac{dx}{x(x^n+1)}$, $n \in N$, take x^n common & put $1 + x^{-n} = t$.

(ii) $\int \frac{dx}{x^2(x^n+1)^{(n-1)/n}}$, $n \in N$, take x^n common & put $1 + x^{-n} = t^n$

(iii) $\int \frac{dx}{x^n(1+x^n)^{1/n}}$, take x^n common and put $1 + x^{-n} = t^n$.

Illustration 16 **:** Evaluate : $\int \frac{dx}{x^n(1+x^n)^{1/n}}$

Solution : Let $I = \int \frac{dx}{x^n(1+x^n)^{1/n}} = \int \frac{dx}{x^{n+1}\left(1+\frac{1}{x^n}\right)^{1/n}}$

Put $1 + \frac{1}{x^n} = t^n$, then $\frac{1}{x^{n+1}}dx = -t^{n-1}dt$

$I = -\int \frac{t^{n-1}dt}{t} = -\int t^{n-2}dt = -\frac{t^{n-1}}{n-1} + c = \frac{-1}{n-1}\left(1+\frac{1}{x^n}\right)^{\frac{n-1}{n}} + c$ **Ans.**

Do yourself -9 :

(i) Evaluate : $\int \frac{dx}{x(x^2+1)}$ **(ii)** Evaluate : $\int \frac{dx}{x^2(x^3+1)^{2/3}}$ **(iii)** Evaluate : $\int \frac{dx}{x^3(x^3+1)^{1/3}}$

MATHEMATICS

PART-I (SUBS)

TRIGONOMETRY

CHAPTER :- Exponential and Trigonometrical Function of Complex Arguments

(1) If z be of the form $x+iy$, we define the exponential and trigonometrical functions by the following equations

$$e^{z} = 1 + z + \frac{z^2}{\underline{|2}} + \frac{z^3}{\underline{|3}} + \cdots\cdots$$

$$\sin z = z - \frac{z^3}{\underline{|3}} + \frac{z^5}{\underline{|5}} - \cdots\cdots$$

$$\cos z = 1 - \frac{z^2}{\underline{|2}} + \frac{z^4}{\underline{|4}} - \cdots\cdots$$

$$a^{z} = e^{z \log a}$$

(2) Exponential Values of Sine and Cosine (Euler's Theorem)

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

(3) Evaluation of $\log(\alpha+i\beta)$ or

Express $\log(\alpha+i\beta)$ in the form $A+iB$ or

Prove that $\log(\alpha+i\beta) = \frac{1}{2}\log(\alpha^2+\beta^2) + i\tan^{-1}\frac{\beta}{\alpha}$

ANS:- Let $\alpha = r\cos\theta$, $\beta = r\sin\theta$

$$\therefore\ r = \sqrt{\alpha^2+\beta^2},\quad \theta = \tan^{-1}\frac{\beta}{\alpha}$$

Now, $\alpha + i\beta = r\cos\theta + i r\sin\theta$

$$= r\left[\frac{e^{i\theta}+e^{-i\theta}}{2} + i\,\frac{e^{i\theta}-e^{-i\theta}}{2i}\right]$$

$$= r.\left[\frac{e^{i\theta}+e^{-i\theta}+e^{i\theta}-e^{-i\theta}}{2}\right]$$

$$= r.e^{i\theta}$$

$$= r\,e^{i\theta}\cdot e^{2n\pi i},\ \text{as}\ e^{2n\pi i} = 1$$

$$\alpha + i\beta = r \cdot e^{i(2n\pi + \theta)}$$

Taking log on both side, we get.

$$\log(\alpha + i\beta) = \log r + \log e^{i(2n\pi + \theta)}$$

$$= \log\sqrt{\alpha^2+\beta^2} + i(2n\pi + \theta)$$

$$\log(\alpha+i\beta) = \frac{1}{2}\log(\alpha^2+\beta^2) + i\left(2n\pi + \tan^{-1}\frac{\beta}{\alpha}\right)$$

Putting $n=0$ in the above result the principal Valeu of $\log(\alpha+i\beta)$ is Obtained

$$\therefore \log(\alpha+i\beta) = \frac{1}{2}\log(\alpha^2+\beta^2) + i\tan^{-1}\frac{\beta}{\alpha}$$

Worked out Problems

Q(1) If $\tan(x+iy) = u+iv$, Prove that $u^2+v^2+2u\cot 2x = 1$

Ans:- $\tan(x+iy) = u+iv$

Writing $-i$ for i, we get

$$\therefore \tan(x-iy) = u-iv$$

Now, $2x = (x+iy)+(x-iy)$

$$= \tan^{-1}(u+iv) + \tan^{-1}(u-iv)$$

$$= \tan^{-1}\frac{(u+iv)+(u-iv)}{1-(u+iv)(u-iv)}$$

$$= \tan^{-1}\frac{2u}{1-(u^2-i^2v^2)}$$

$$2x = \tan^{-1}\frac{2u}{1-u^2-v^2}$$

$$\therefore \tan 2x = \frac{2u}{1-u^2-v^2}$$

LOGY SUBS

or, $\frac{1}{\cot 2x} = \frac{2u}{1-u^2-v^2}$

or $2u\cdot\cot 2x = 1-u^2-v^2$

or $u^2+v^2+2u\cot 2x = 1$ //

Q(2) If $A+iB = \log(m+in)$, Show that $\tan\theta = \frac{n}{m}$ and $2A = \log(n^2+m^2)$

Ans:- We have,

$$A+iB = \log(m+in)$$

$$= \log\sqrt{m^2+n^2} + i\tan^{-1}\frac{n}{m}$$

Equating real and imaginary Part, we get

$\therefore A = \log\sqrt{n^2+m^2}$ and $B = \tan^{-1}\frac{n}{m}$

or $2A = \log(n^2+m^2)$ and $\tan B = \frac{n}{m}$ //

Q(3) Prove that $\tan\left\{i\log\frac{a-ib}{a+ib}\right\} = \frac{2ab}{a^2-b^2}$ where a and b are two real quantities.

Ans:- Let $a = r\cos\theta$ and $b = r\sin\theta$.

$\therefore r = \sqrt{a^2+b^2}$ and $\tan\theta = \frac{b}{a}$

$$L.H.S. = \tan\left\{i\log\frac{a-ib}{a+ib}\right\}$$

$$= \tan\left\{i\log\frac{r(\cos\theta - i\sin\theta)}{r(\cos\theta+i\sin\theta)}\right\}$$

$$= \tan\left\{i\log\frac{e^{-i\theta}}{e^{i\theta}}\right\}$$

$= \tan\{i \log e^{-2i\theta}\}$

$= \tan\{i \times -2i\theta . \log e\}$

$= \tan\{-2i^2\theta \times 1\}$

$= \tan\{2\theta \times 1\}$

$= \tan 2\theta$

$= \frac{2\tan\theta}{1-\tan^2\theta}$

$= \frac{2\frac{b}{a}}{1-\frac{b^2}{a^2}}$

$= \frac{2b}{a} \times \frac{a^2}{a^2-b^2}$

$= \frac{2ab}{a^2-b^2}$ = R.H.S. Proved.

Q(4) If $\tan\log(x+iy) = a+ib$, where $a^2+b^2 \neq 1$

Prove that $\tan\{\log(x^2+y^2)\} = \frac{2a}{1-a^2-b^2}$

Ans:- We have, $\tan\log(x+iy) = a+ib$

$\therefore \tan\log(x-iy) = a-ib$

Now, $\tan\{\log(x^2+y^2)\} = \tan\{\log(x+iy)(x-iy)\}$

$= \tan\{\log(x+iy) + \log(x-iy)\}$

$$= \frac{\tan\{\log(x+iy)\} + \tan\{\log(x-iy)\}}{1 - \tan\log(x+iy)\tan\log(x-iy)}$$

$$= \frac{a+ib+a-ib}{1-(a+ib)(a-ib)}$$

$$= \frac{2a}{1-(a^2-i^2b^2)}$$

$$\tan\{\log(x^2+y^2)\} = \frac{2a}{1-a^2-b^2}$$ //

Q(5) Prove that $i^i = e^{-(4n+1)\frac{\pi}{2}}$

Ans:- We know that

$$i = e^{\frac{i\pi}{2}} = e^{\frac{i\pi}{2}} \cdot e^{2n\pi i} = e^{\frac{i\pi}{2} + 2n\pi i}$$

$$\therefore i^i = \left\{e^{\frac{i\pi}{2}(4n+1)}\right\}^i$$

$$= e^{-\frac{\pi}{2}(4n+1)}$$

Q(6) If $i^{\alpha+i\beta} = \alpha+i\beta$ Prove that $\alpha^2+\beta^2 = e^{-(4n+1)\pi\beta}$

Ans:- $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{\frac{\pi i}{2}}$

Now, $i^{\alpha+i\beta} = \alpha+i\beta$

$$\Rightarrow \left(e^{\frac{\pi i}{2}}\right)^{\alpha+i\beta} = \alpha+i\beta$$

$$\Rightarrow e^{i\left(2n\pi+\frac{\pi}{2}\right)(\alpha+i\beta)} = \alpha+i\beta$$

$$\Rightarrow e^{i\alpha\left(2n\pi+\frac{\pi}{2}\right) - \beta\left(2n\pi+\frac{\pi}{2}\right)} = \alpha+i\beta$$

$$\Rightarrow e^{-\beta(2n\pi+\frac{\pi}{2})} \cdot e^{i\alpha(2n\pi+\frac{\pi}{2})} = \alpha + i\beta$$ ⑥

$$\Rightarrow e^{-\beta(2n\pi+\frac{\pi}{2})}\left[\cos(2n\pi+\tfrac{\pi}{2})\alpha + i\sin(2n\pi+\tfrac{\pi}{2})\alpha\right] = \alpha + i\beta$$

Equating real and imaginary part, we get

$$e^{-\beta(2n\pi+\frac{\pi}{2})} \cdot \cos(2n\pi+\tfrac{\pi}{2})\alpha = \alpha \quad \text{--- (1)}$$

and $e^{-\beta(2n\pi+\frac{\pi}{2})} \cdot \sin(2n\pi+\frac{\pi}{2})\alpha = \beta$ --- (2)

Squaring and adding, we get

$$\therefore \alpha^2+\beta^2 = e^{-2\beta(2n\pi+\frac{\pi}{2})}$$

$$\therefore \alpha^2+\beta^2 = e^{-(4n+1)\pi\beta} \quad //$$

Q⑦ Prove that $\sin^{-1}(i) = 2n\pi - i\log(\sqrt{2}-1)$

Ans:- Put $i = \sin\theta$ then $\sin^{-1}(i) = \theta$

Now, $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-i^2} = \sqrt{2}$

$$\therefore \log(\sqrt{2}-1) = \log(\cos\theta + i^2)$$

$$= \log(\cos\theta + i\sin\theta)$$

$$= \log e^{i\theta} = \log e^{i\theta - 2n\pi i}$$

$$= i\theta - 2n\pi i$$

$$\therefore i\theta = 2n\pi i + \log(\sqrt{2}-1)$$

$$\theta = 2n\pi + \frac{1}{i}\log(\sqrt{2}-1)$$

$\therefore \sin^{-1}(i) = 2n\pi - i\log(\sqrt{2}-1)$ //

Q ⑤ Reduce $(\alpha+i\beta)^{x+iy}$ to the form $A+iB$. (8)

Ans:- By definition, $a^x = e^{x\log a}$

Similarly, $(\alpha+i\beta)^{x+iy} = e^{(x+iy)\log(\alpha+i\beta)}$

We know that $\log(\alpha+i\beta) = \frac{1}{2}\log(\alpha^2+\beta^2) + i\left(2n\pi + \tan^{-1}\frac{\beta}{\alpha}\right)$

$$\therefore (x+iy)\log(\alpha+i\beta) = (x+iy)\left[\frac{1}{2}\log(\alpha^2+\beta^2) + i\left(2n\pi + \tan^{-1}\frac{\beta}{\alpha}\right)\right]$$

$$= \left[\frac{x}{2}\log(\alpha^2+\beta^2) - y\left(2n\pi + \tan^{-1}\frac{\beta}{\alpha}\right)\right] + i\left[x\left(2n\pi + \tan^{-1}\frac{\beta}{\alpha}\right) + \frac{y}{2}\log(\alpha^2+\beta^2)\right]$$

$$= p + iq \quad (\text{say})$$

Then, $(\alpha+i\beta)^{x+iy} = e^{p+iq} = e^p \cdot e^{iq}$

$$= e^p(\cos q + i\sin q)$$

$$= e^{\frac{x}{2}\log(\alpha^2+\beta^2) - y\left(2n\pi + \tan^{-1}\frac{\beta}{\alpha}\right)} \times \left[\cos\left\{x\left(2n\pi + \tan^{-1}\frac{\beta}{\alpha}\right) + \frac{y}{2}\log(\alpha^2+\beta^2)\right\} + i\sin\left\{x\left(2n\pi + \tan^{-1}\frac{\beta}{\alpha}\right) + \frac{y}{2}\log(\alpha^2+\beta^2)\right\}\right]$$

This is the form $A+iB$.

Q ⑥ Find the general value of $\log_e(-3)$

Ans:- $\log_e(-3) = \log_e(3i^2) = \log_e 3 + \log_e i^2$

$$= \log_e 3 + 2\log_e i = \log_e 3 + 2\left[i\tan^{-1}\frac{1}{0} + 2n\pi i\right]$$

$$= \log_e 3 + 2\left(i\frac{\pi}{2} + 2n\pi i\right) = \log_e 3 + (4n+1)\frac{\pi i}{2}$$